Sampling & Quantization

Dr. Tushar Sandhan

Introduction

Introduction

- Input

- Sampling

- Quantization

Sampling

- Sampling
  - determines spatial resolution
  - space digitization

Sampling

- Sampling
  - determines spatial resolution
  - space digitization

- Image frequency
  - what are freq contents inside an image?
  - is the uniform sampling optimal?
  - is oversampling useful?
  - strive for efficient sampling
    - sampling density
    - data storage, data transmission

Sampling

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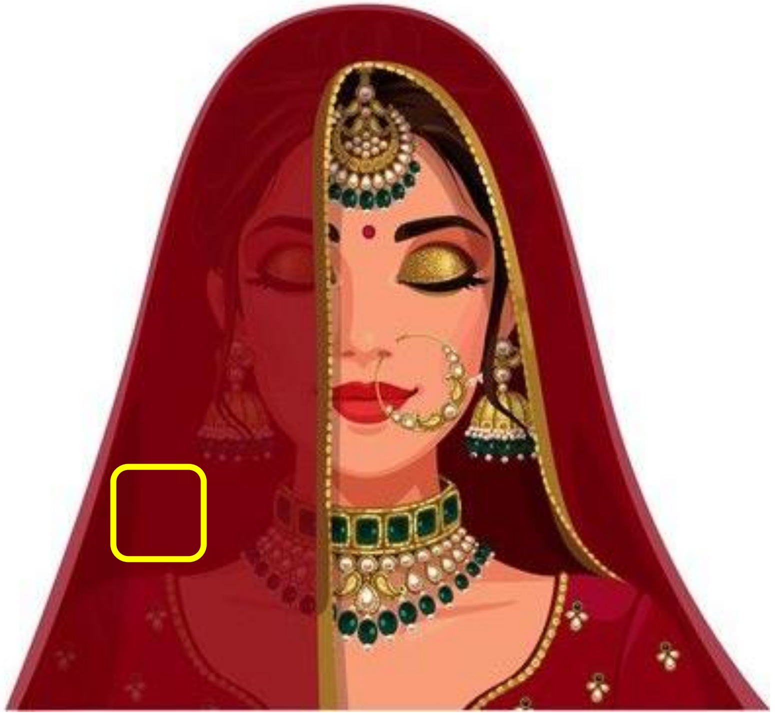
  - determines spatial resolution
  - space digitization



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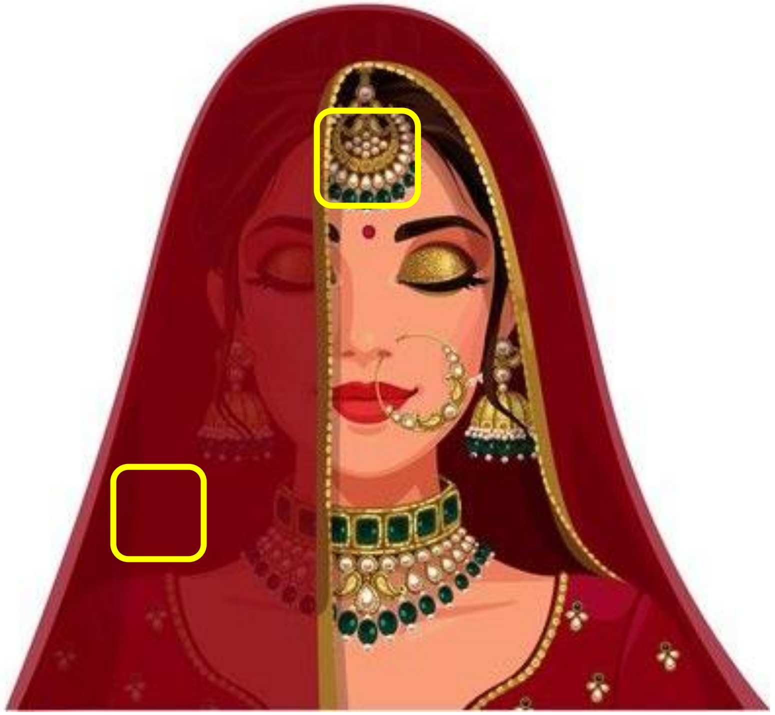

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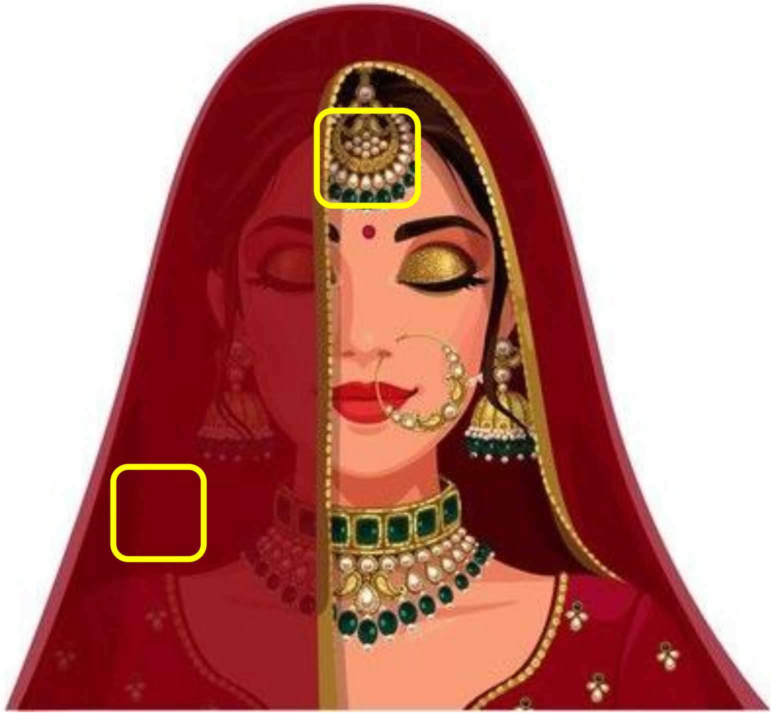

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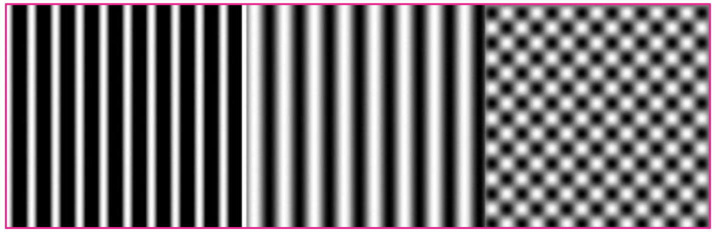
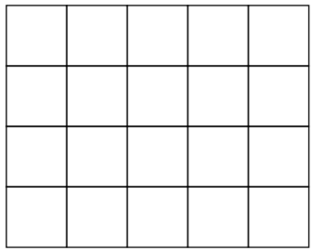
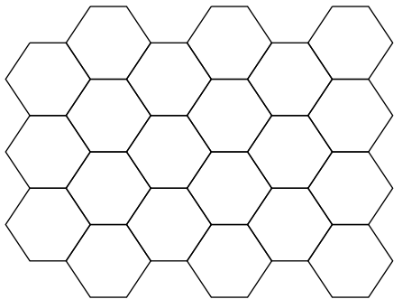
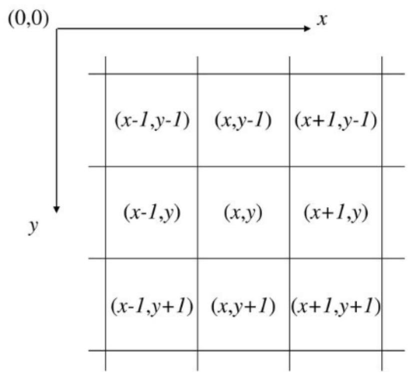


Sampling

- Grid
  - continuous image is digitized at sampling points
  - sampling points ordered in the plane
  - their geometric relation – grid
  - smallest grid point corresponds to – pixel (2D)
  - voxel (3D)







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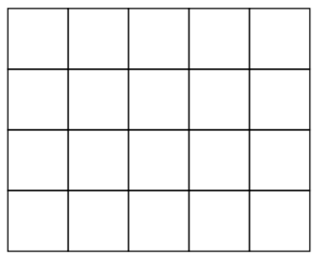
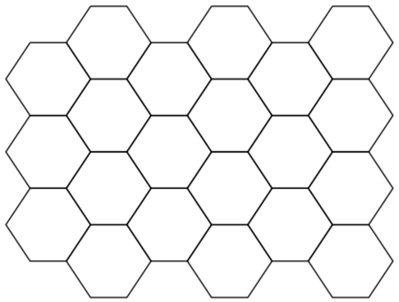
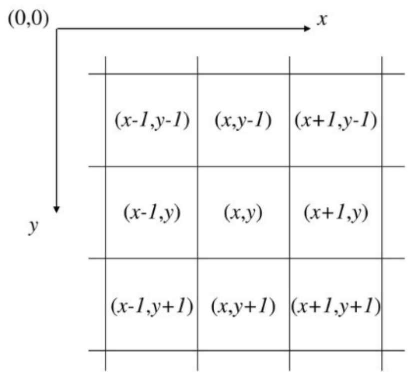




- Neighbourhood



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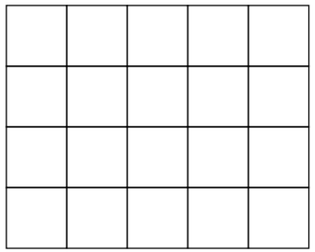

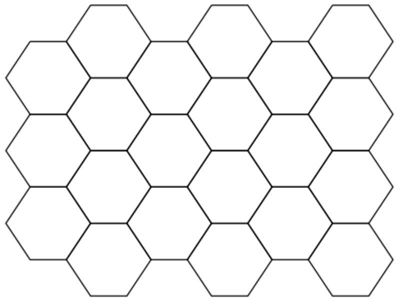

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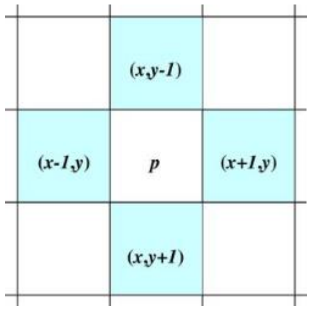
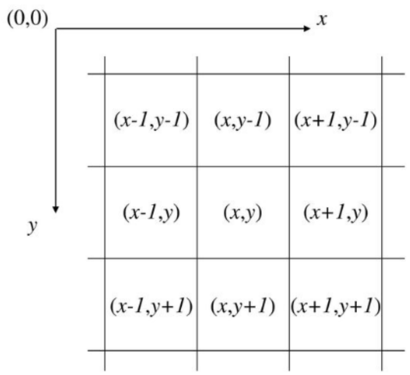


$$N_4(p) = \left\{ \begin{matrix} (x-1,y) \\ (x+1,y) \\ (x,y-1) \\ (x,y+1) \end{matrix} \right\}$$



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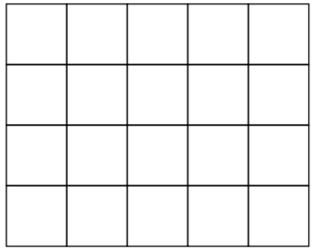

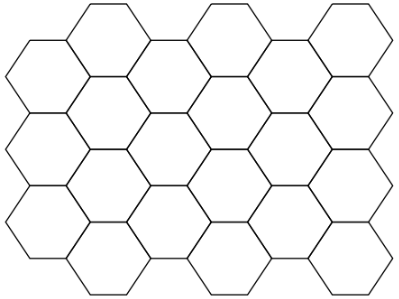

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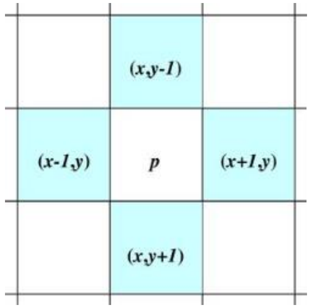

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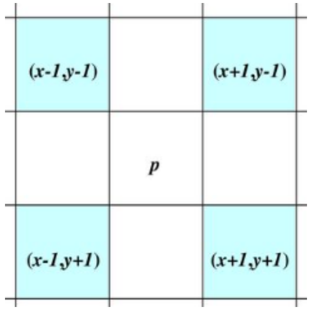
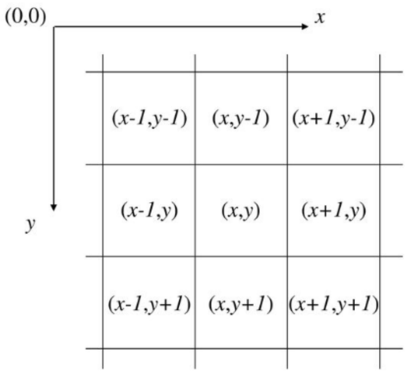


$N_D(p)$



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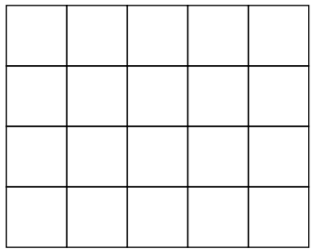

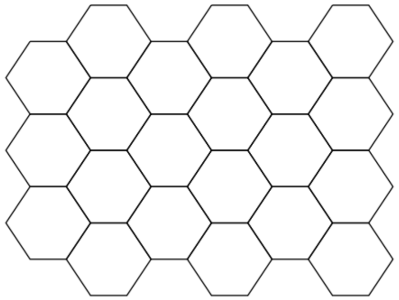

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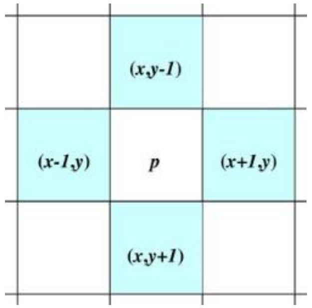

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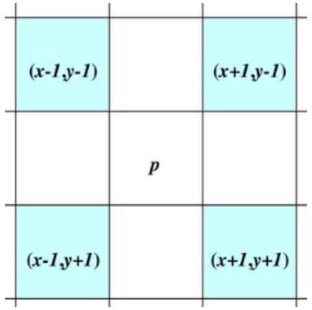

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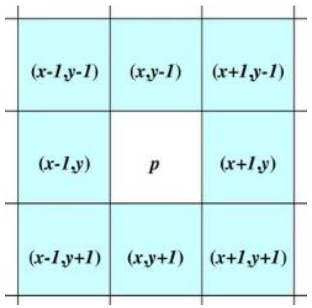
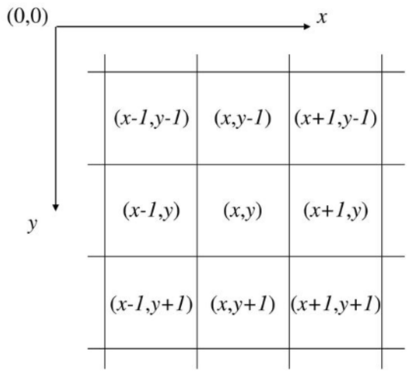
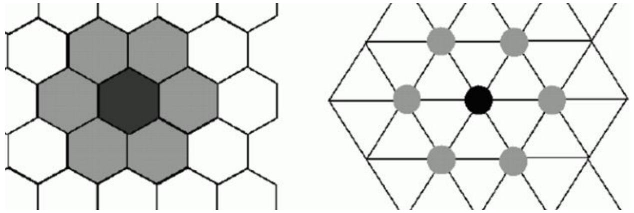


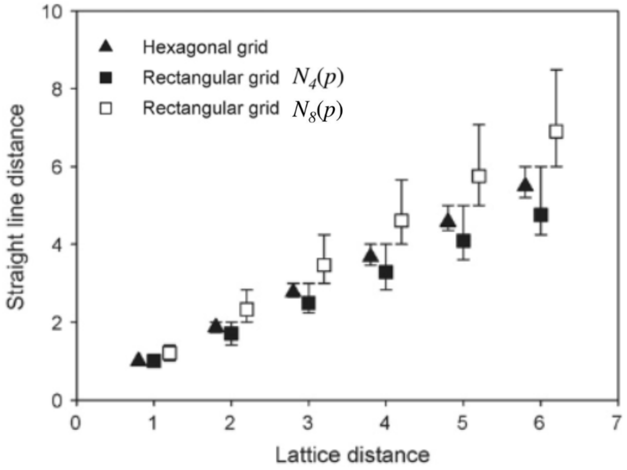
$N_8(p)$



Sampling

- Neighbourhood
  - hexagonal grid
  - neighbour interaction





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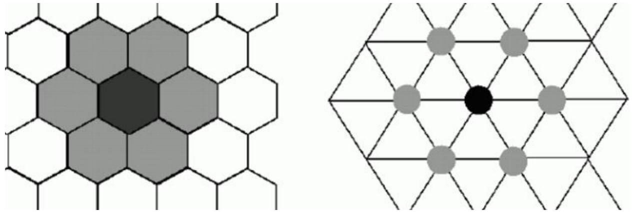

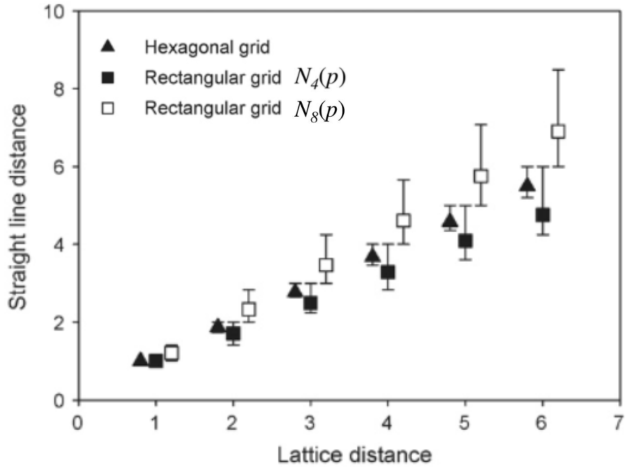

- Neighbour interactions
  - distance, energy, edges, features
  - sq. grid neighbourhood paradox
    - N_4: broken ring encloses
    - N_8: complete ring without enclosure

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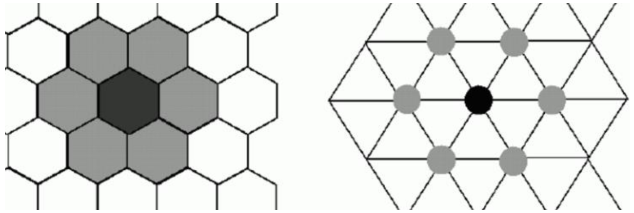

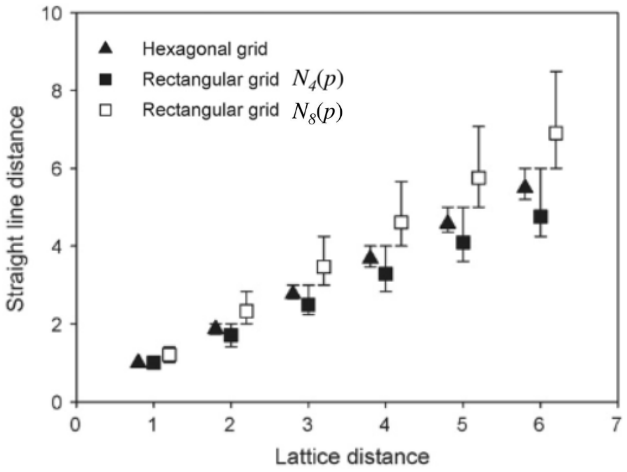

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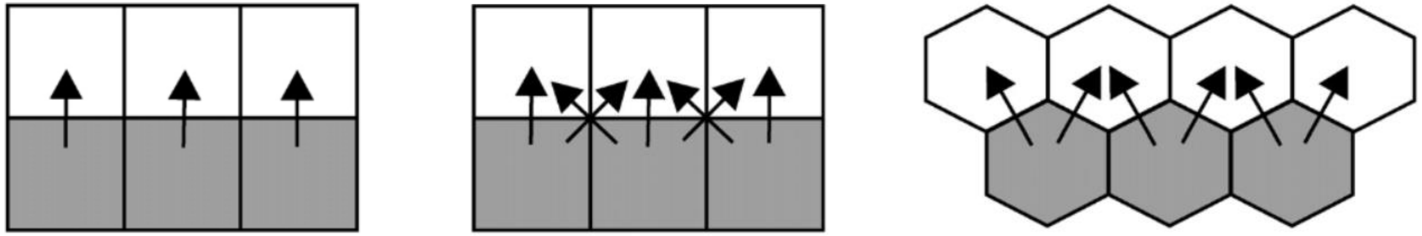

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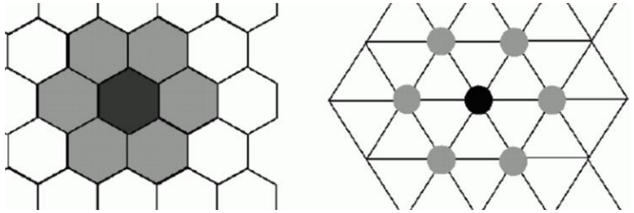

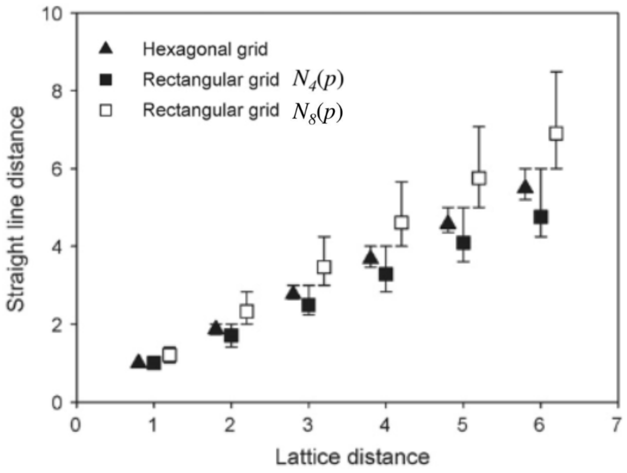

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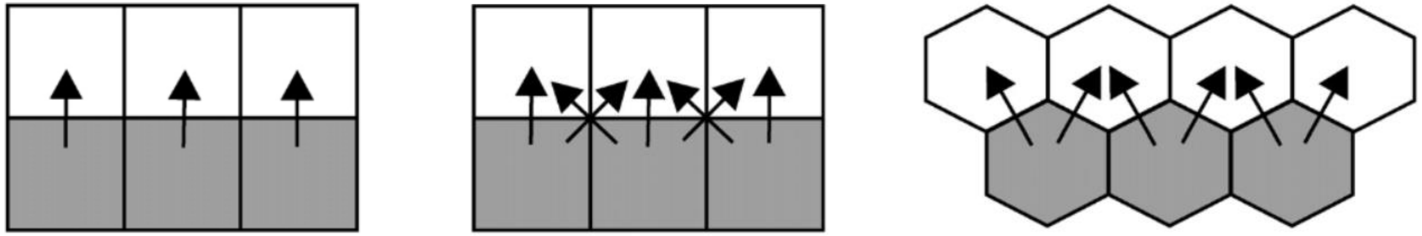
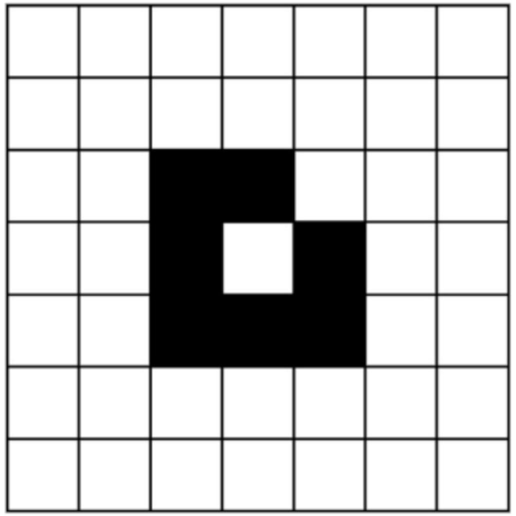
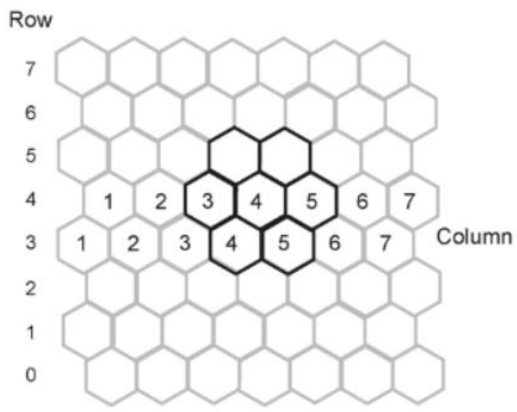




Sampling

- Coordinate system
  - hexagonal grid
  - SHCS: symmetrical hexagonal coordinate system in (C)



(a)

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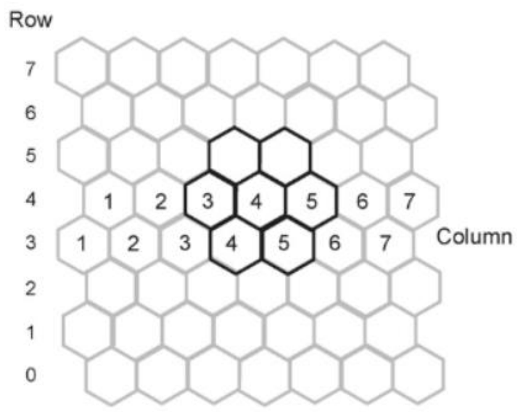

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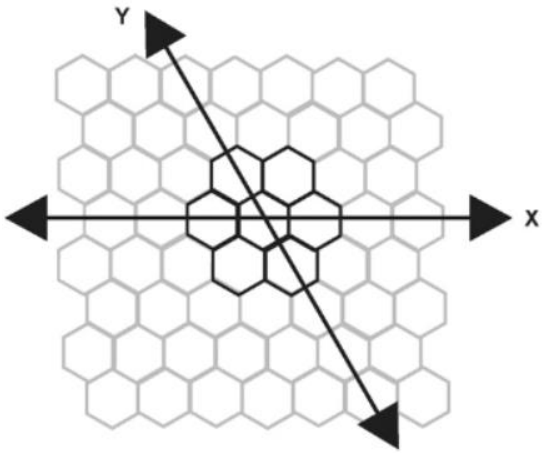

(b)

Sampling

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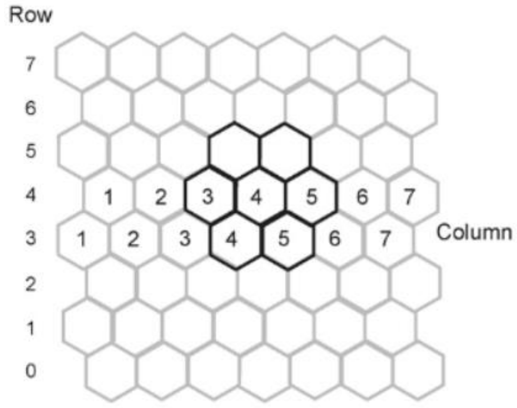

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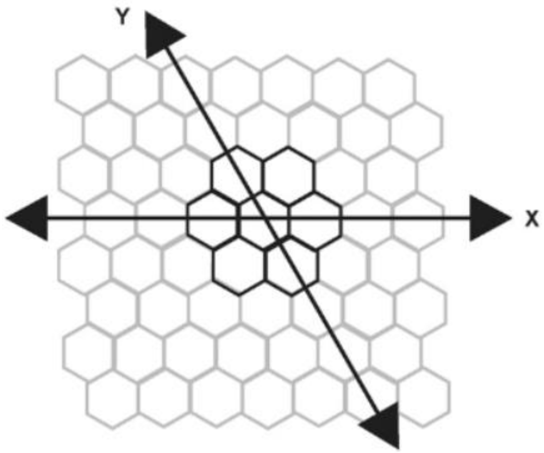

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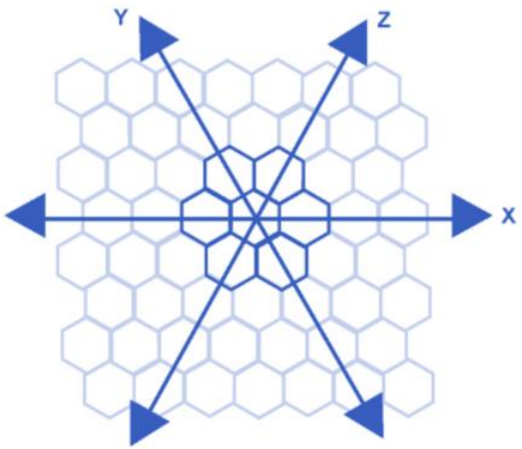

(C)

Sampling

- Coordinate system
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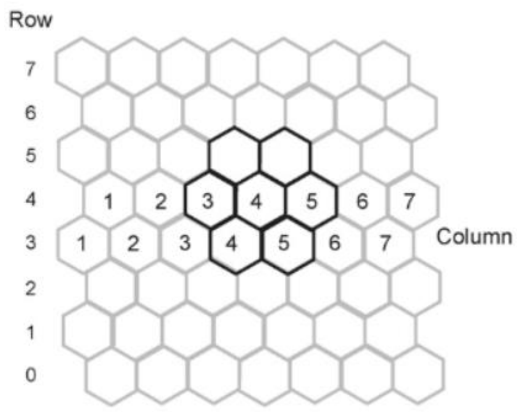
  - SHCS: symmetrical hexagonal coordinate system in (C)

$$\forall(x, y, z) : x + y + z = 0$$

(x_1, y_1, z_1) and (x_2, y_2, z_2) are the co-ordinates of the two hexagons.

$$D_{\text{Eucl.}}[(x_1, y_1, x_1), (x_2, y_2, x_2)] = \sqrt{\tfrac{1}{2}[(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2]}$$

$$D_{\text{Grid}}[(x_1, y_1, x_1), (x_2, y_2, x_2)] = \max(|x_1 - x_2|, |y_1 - y_2|, |z_1 - z_2|)$$



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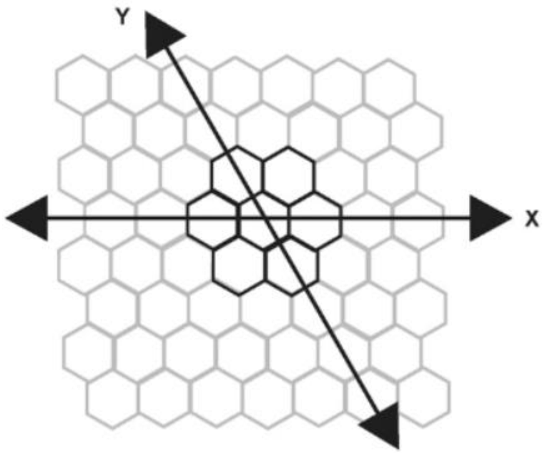

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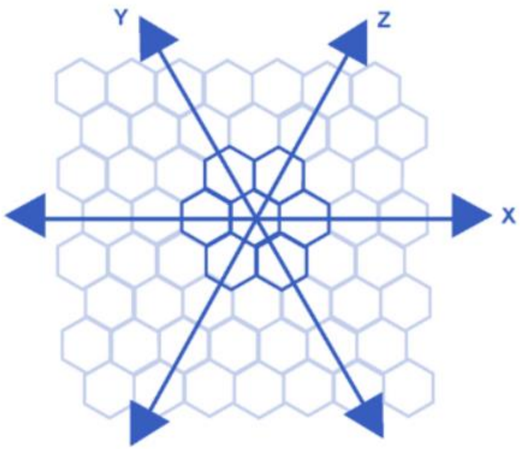

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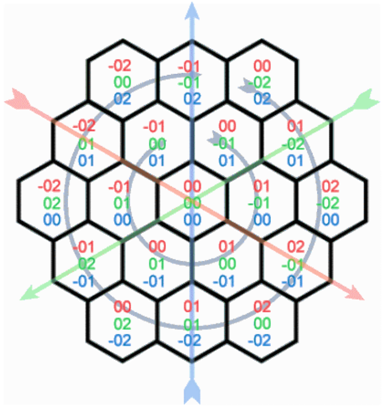

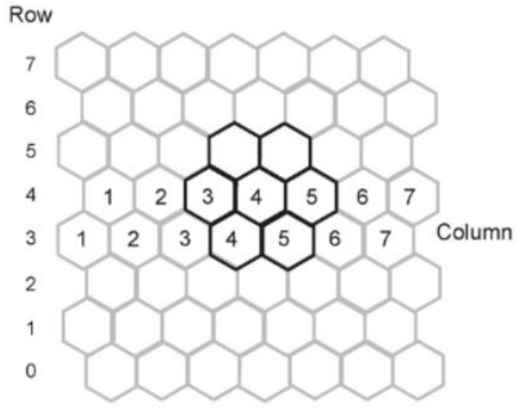

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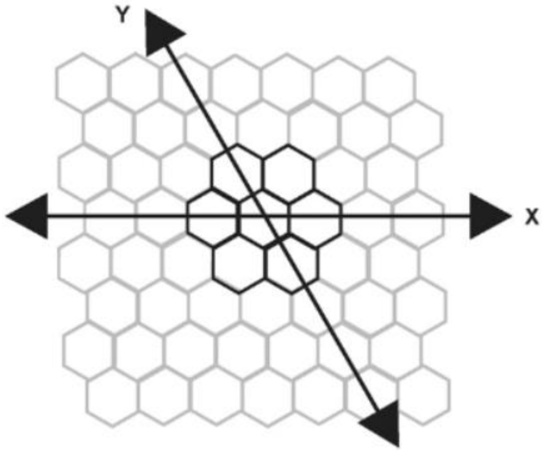

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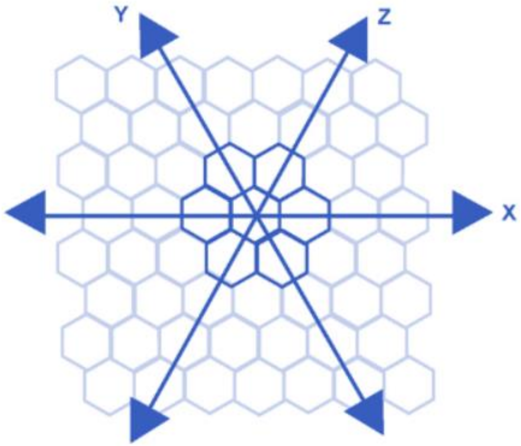

(C)

Sampling

- Hierarchical grids
  - neighbours at finer scale become
    focal cell or centroids for coarser scale
  - smooth out or simplify some grids
    - dynamic grid resolution
  - θ, r can be used to find out current resolution scale

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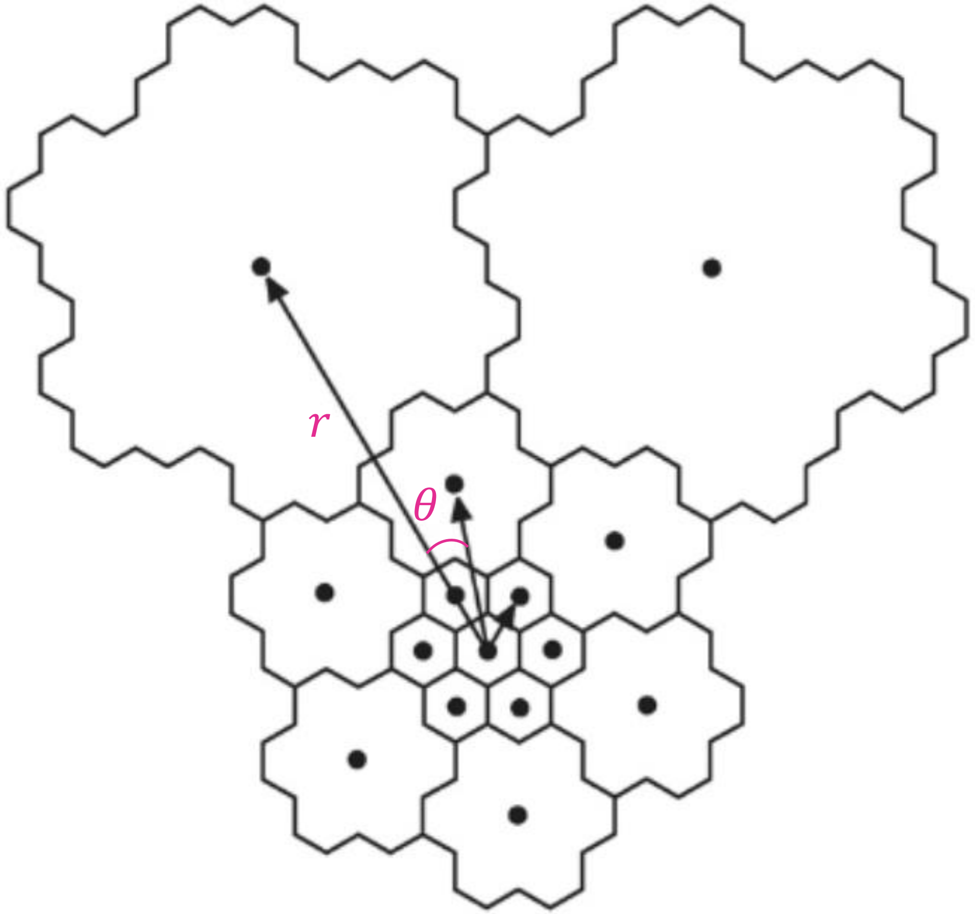

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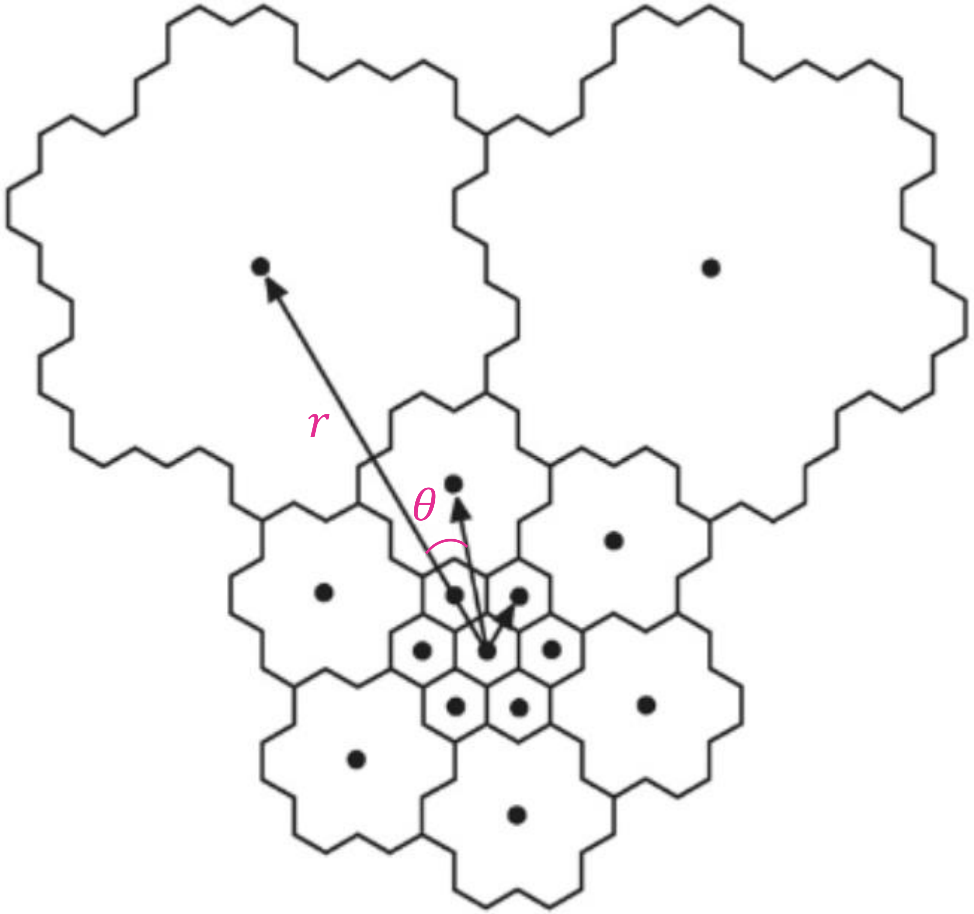

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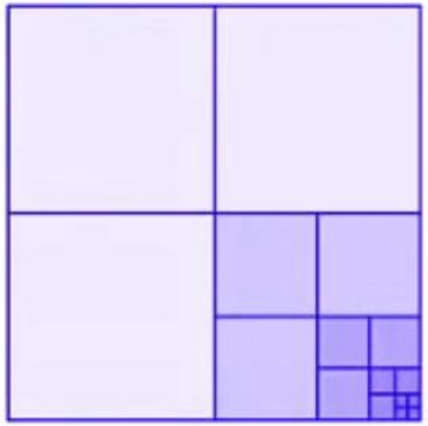

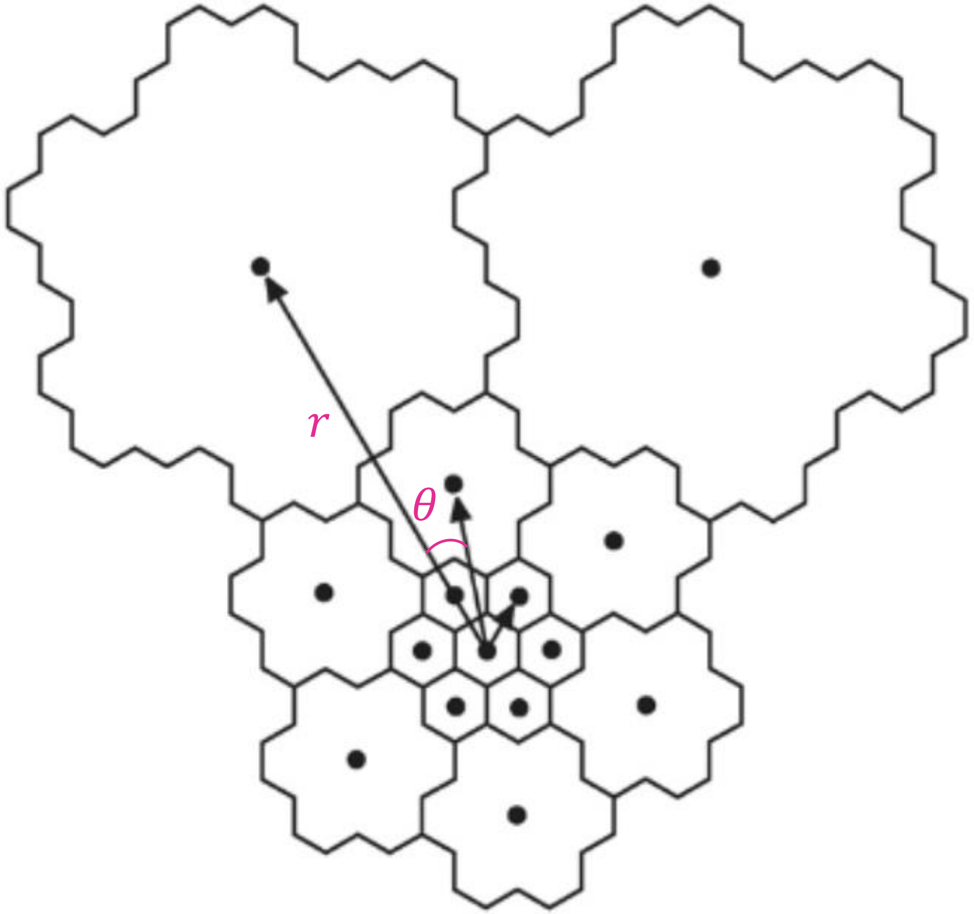

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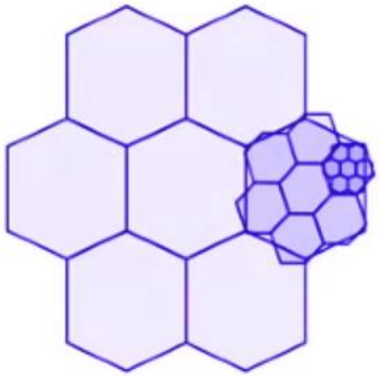

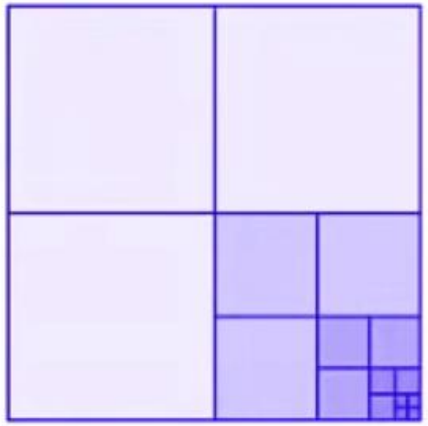

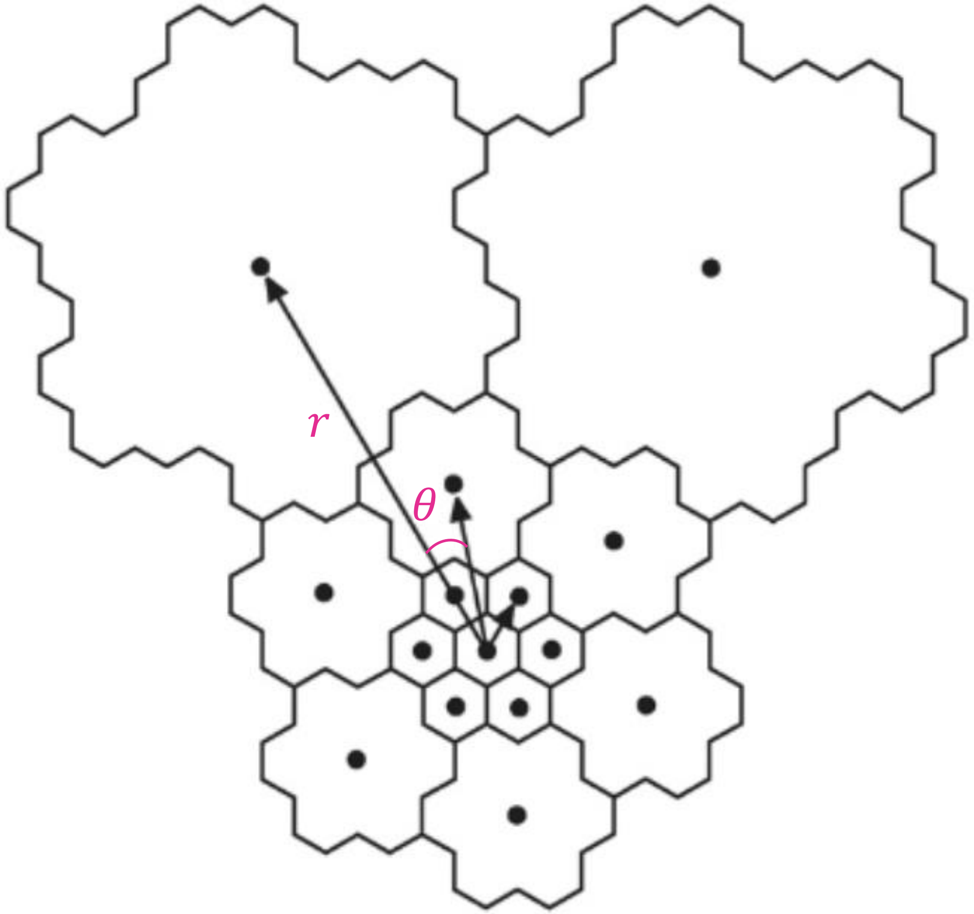

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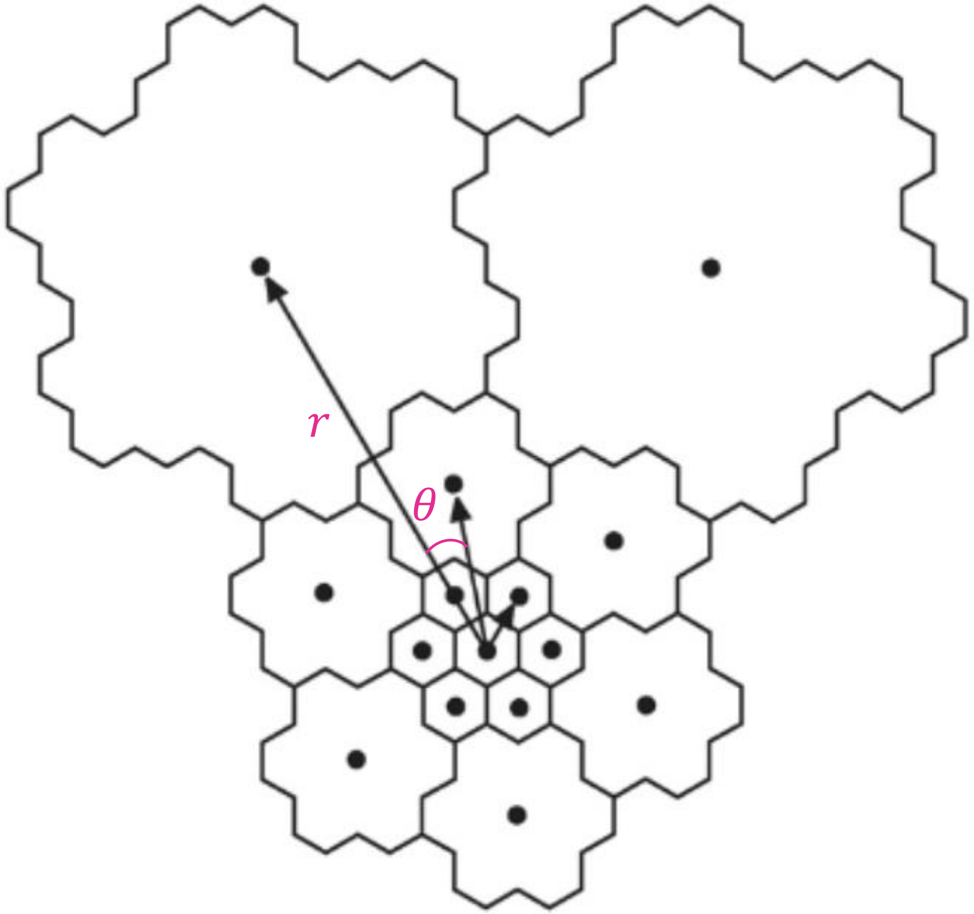
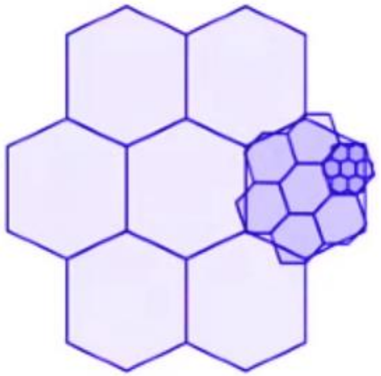
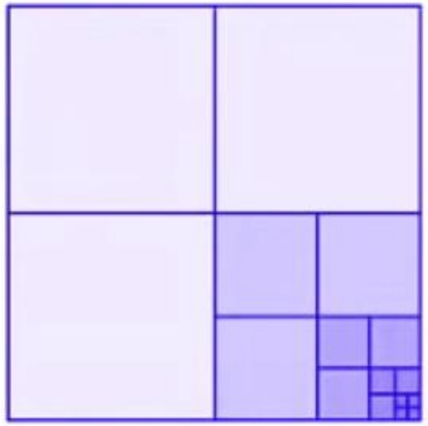




Alternating CW, CCW
19.1^o rotations of 7
children 1/7th the area



Sampling

- Checkerboard effect
  - due to uniform non-optimal square grid sampling

Sampling

- Checkerboard effect
  - due to uniform non-optimal square grid sampling

128x128

64x64

32x32

Sampling

- Aliasing

continuous bandlimited function

$$f(x, y)$$

$$\delta(x, y) = \begin{cases} 1 & \text{if } x = y = 0 \\ 0 & \text{otherwise} \end{cases}$$

$f(x, y)$ is a func of discrete vars x, y

Sampling

- Aliasing

continuous bandlimited function

$$f(x, y)$$

$$\delta(x, y) = \begin{cases} 1 & \text{if } x = y = 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\sum_{x=-\infty}^{\infty} \sum_{y=-\infty}^{\infty} f(x, y)\,\delta(x - x_0, y - y_0) = f(x_0, y_0)$$

$f(x, y)$ is a func of discrete vars x, y

Sampling

- Aliasing

continuous bandlimited function

$$f(x,y)$$

$$comb(x,y,\Delta x,\Delta y) = \sum_m \sum_n \delta(x - m\Delta x, y - n\Delta y)$$

$$\delta(x,y) = \begin{cases} 1 & \text{if } x = y = 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\sum_{x=-\infty}^{\infty} \sum_{y=-\infty}^{\infty} f(x,y)\,\delta(x - x_0, y - y_0) = f(x_0, y_0)$$

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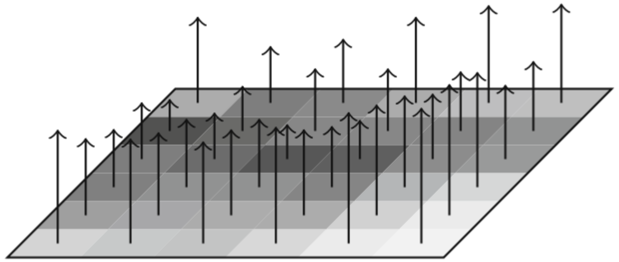
Sampling

- Aliasing

$$comb(x, y, \Delta x, \Delta y) = \sum_m \sum_n \delta(x - m\Delta x, y - n\Delta y)$$

continuous bandlimited function

$f(x, y)$



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$f(x, y)$ is a func of discrete vars x, y

Sampling

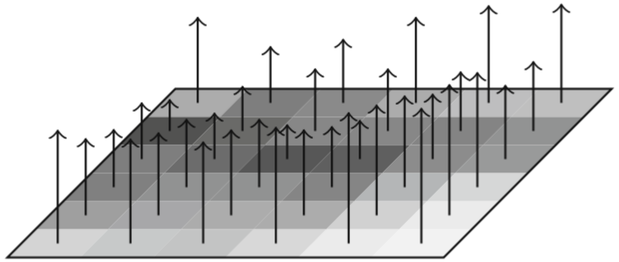
- Aliasing

$$f_s(x,y) = f(x,y)\, comb(x,y,\Delta x,\Delta y)$$

$$comb(x,y,\Delta x,\Delta y) = \sum_m \sum_n \delta(x - m\Delta x, y - n\Delta y)$$

continuous bandlimited function

$f(x,y)$



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Sampling

- Aliasing

continuous bandlimited function

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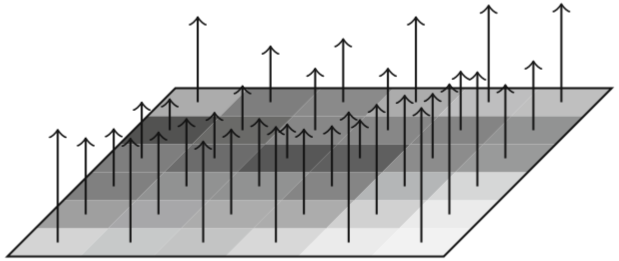
$f(x,y)$



$$comb(x,y,\Delta x,\Delta y) = \sum_m \sum_n \delta(x - m\Delta x, y - n\Delta y)$$

$$F_s(\omega_x,\omega_y) = F(\omega_x,\omega_y) * \omega_{x_s}\omega_{y_s} \sum_p \sum_q \delta(\omega_x - p\omega_{x_s}, \omega_y - q\omega_{y_s})$$
$$= \omega_{x_s}\omega_{y_s} \sum_p \sum_q F(\omega_x - p\omega_{x_s}, \omega_y - q\omega_{y_s})$$

$$\delta(x,y) = \begin{cases} 1 & \text{if } x = y = 0 \\ 0 & \text{otherwise} \end{cases}$$

where $\omega_{xs} = \frac{2\pi}{\Delta x}$, and $\omega_{ys} = \frac{2\pi}{\Delta y}$

$$\sum_{x=-\infty}^{\infty} \sum_{y=-\infty}^{\infty} f(x,y)\,\delta(x - x_0, y - y_0) = f(x_0, y_0)$$

$f(x,y)$ is a func of discrete vars x, y

Sampling

- Aliasing

$$f_s(x,y) = f(x,y)\, comb(x,y,\Delta x,\Delta y)$$

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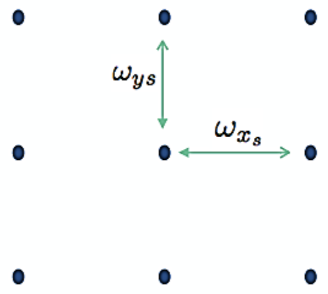
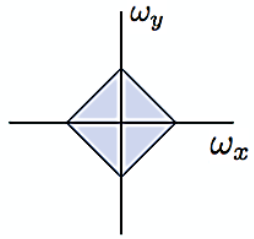
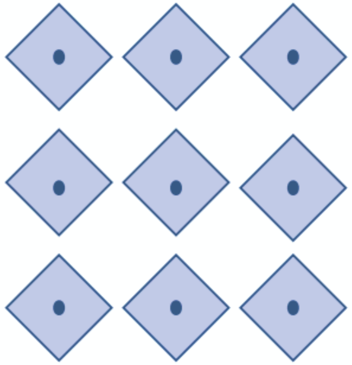


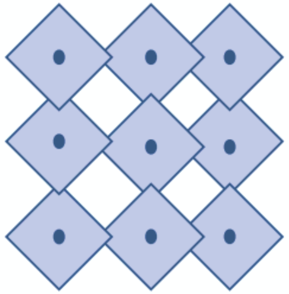
$comb(\omega_x,\omega_y)$



$F(\omega_x,\omega_y)$



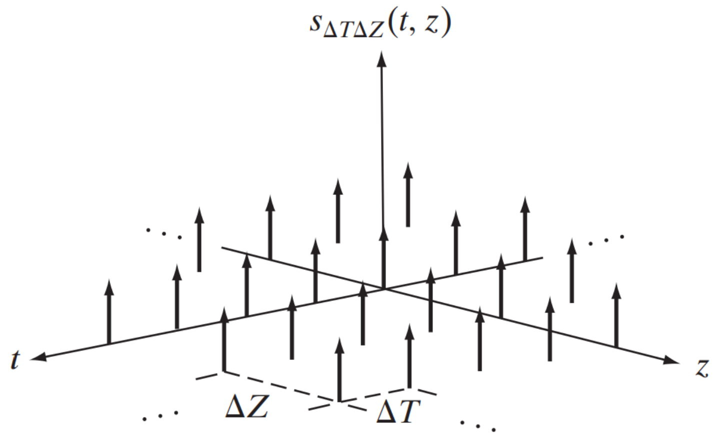
$F_s(\omega_x,\omega_y)$



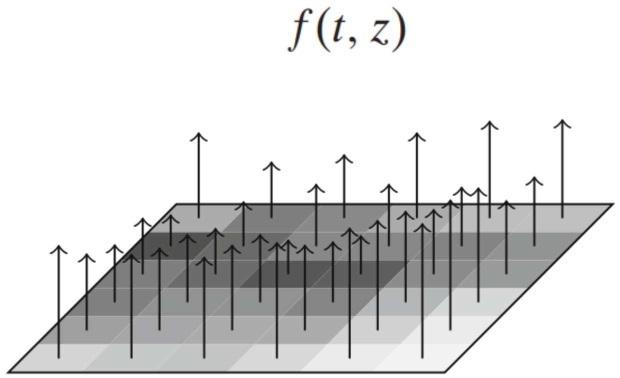
$F_s(\omega_x,\omega_y)$

Sampling

- Sampling theorem
  - $f(t, z)$ can be recovered fully with zero error from its samples
  - iff grid is ‘sufficiently’ dense
  - just change of variables to simplify notations



continuous bandlimited function



$$s_{\Delta T \Delta Z}(t, z) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \delta(t - m\Delta T, z - n\Delta Z)$$

$$F(\mu, \nu) = 0 \quad \text{for } |\mu| \geq \mu_{\max} \text{ and } |\nu| \geq \nu_{\max}$$

… band limits

Sampling

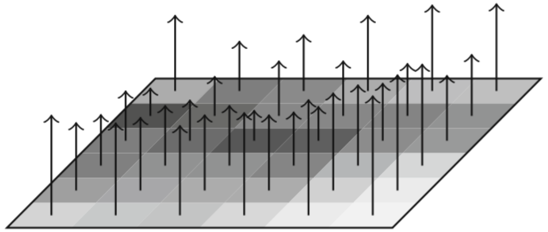
- Sampling theorem
  - $f(t,z)$ can be recovered fully with zero error from its samples
  - No info is lost in the image if it is obtained via sampling at rates greater than twice the max freq content of $f(t,z)$ in both μ, v directions.

continuous bandlimited function

Image $\leftarrow$ $f(t, z)$

$$s_{\Delta T \Delta Z}(t, z) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \delta(t - m\Delta T, z - n\Delta Z)$$

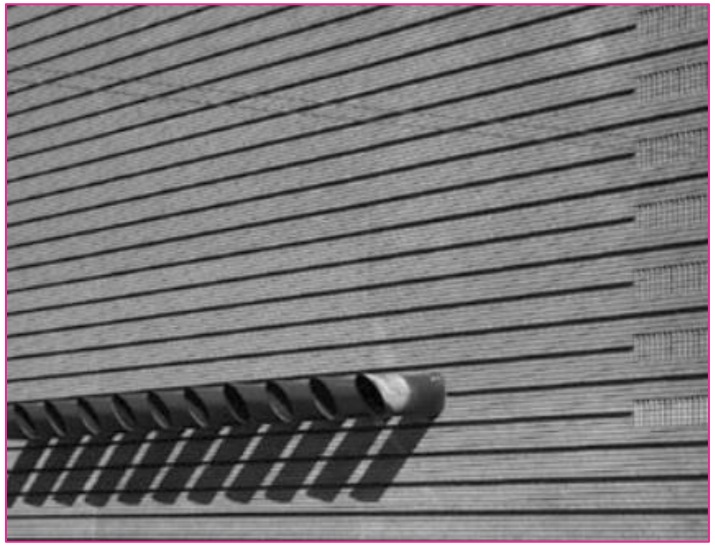
$$F(\mu, \nu) = 0 \quad \text{for} \quad |\mu| \geq \mu_{\max} \quad \text{and} \quad |\nu| \geq \nu_{\max}$$ … band limits



$$\boxed{\frac{1}{\Delta T} > 2\mu_{\max} \qquad \frac{1}{\Delta Z} > 2\nu_{\max}}$$

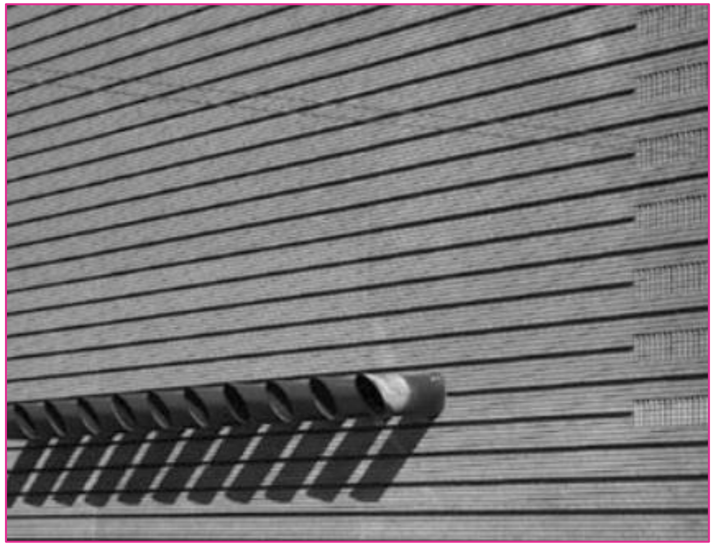
Sampling

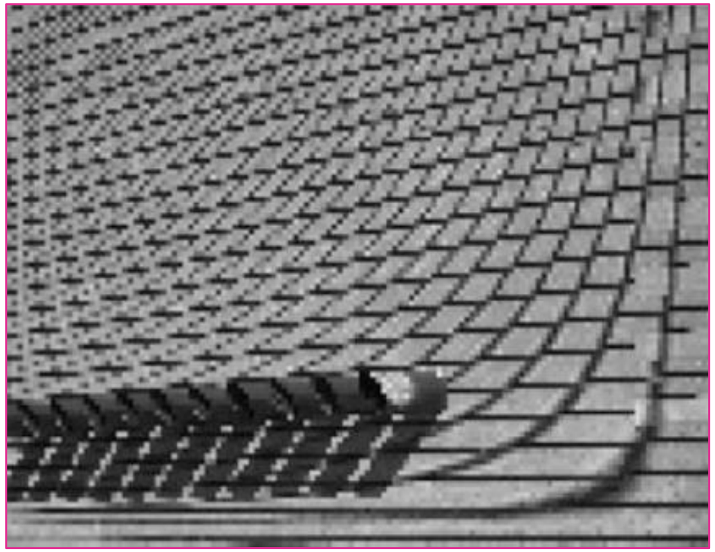
- Erroneous effects
  - square grid sampling



Sampling

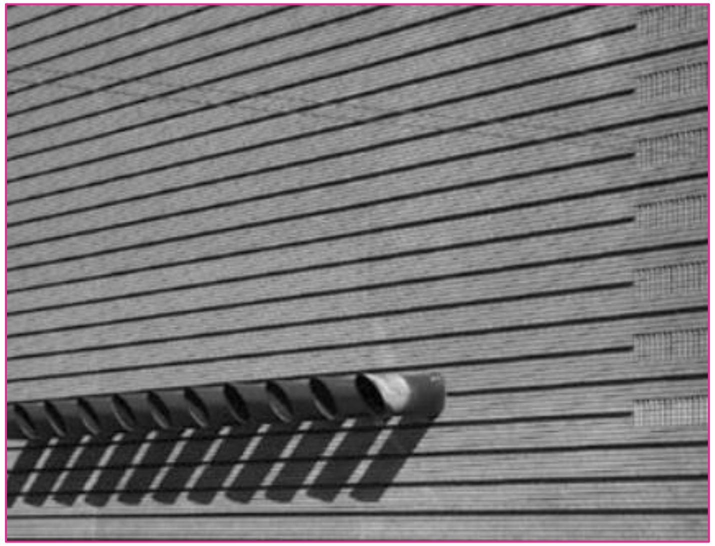
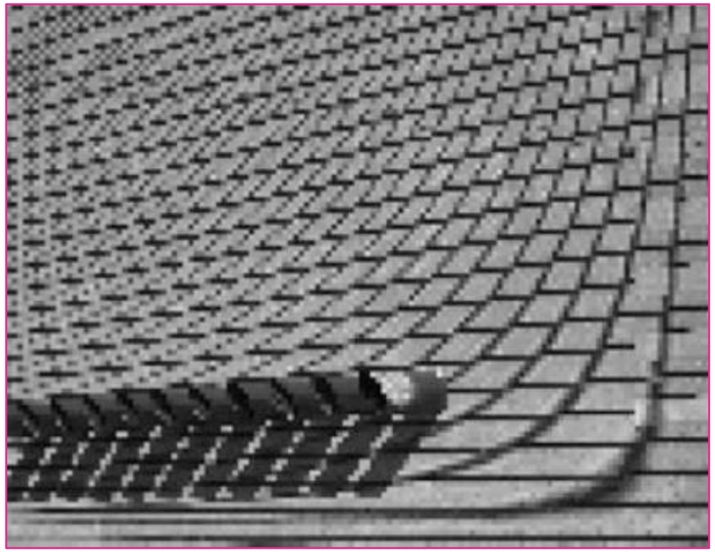
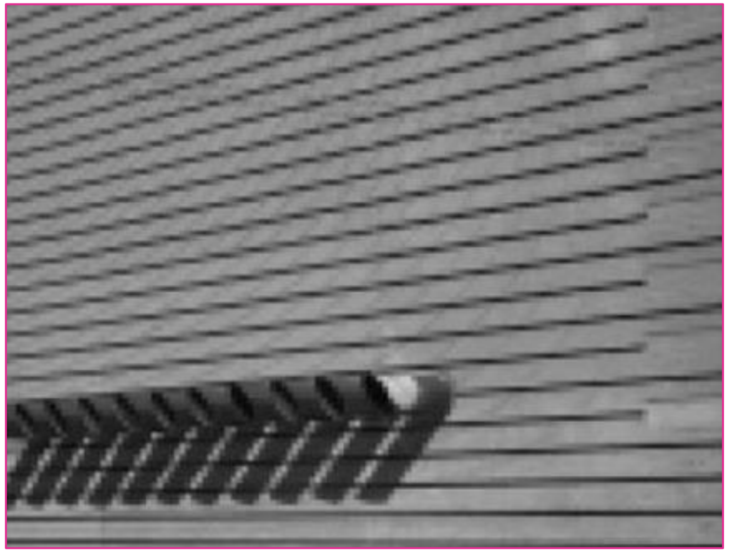
- Erroneous effects
  - square grid sampling





Sampling

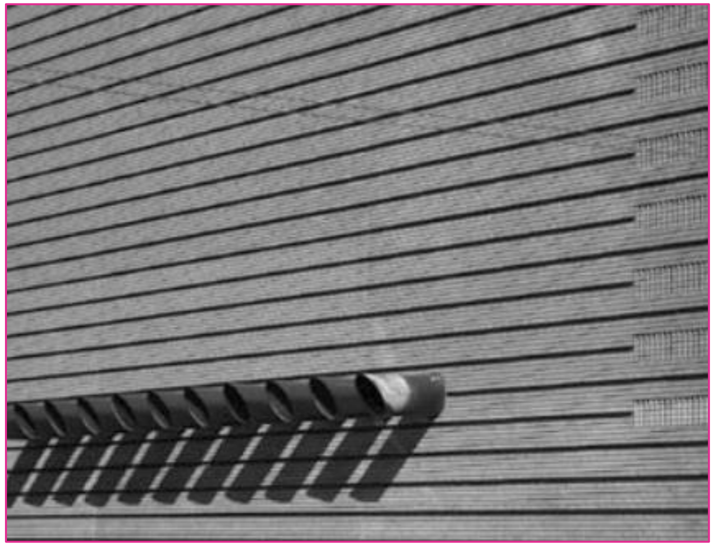
- Erroneous effects
  - square grid sampling

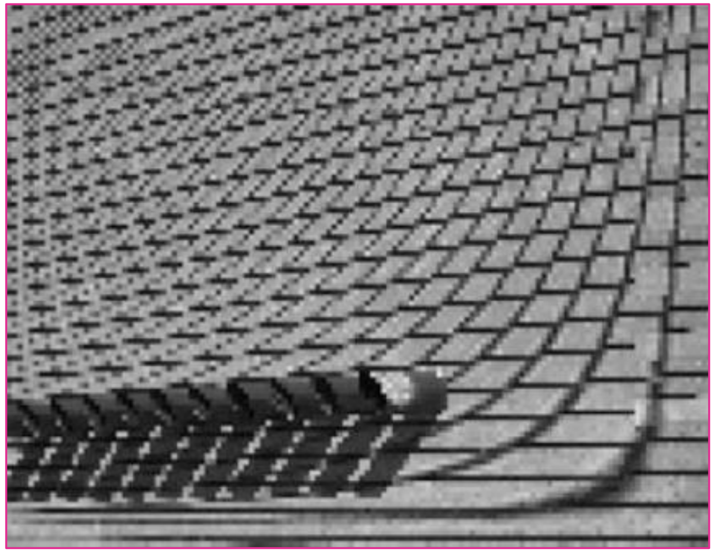
Sampling

- Erroneous effects
  - square grid sampling

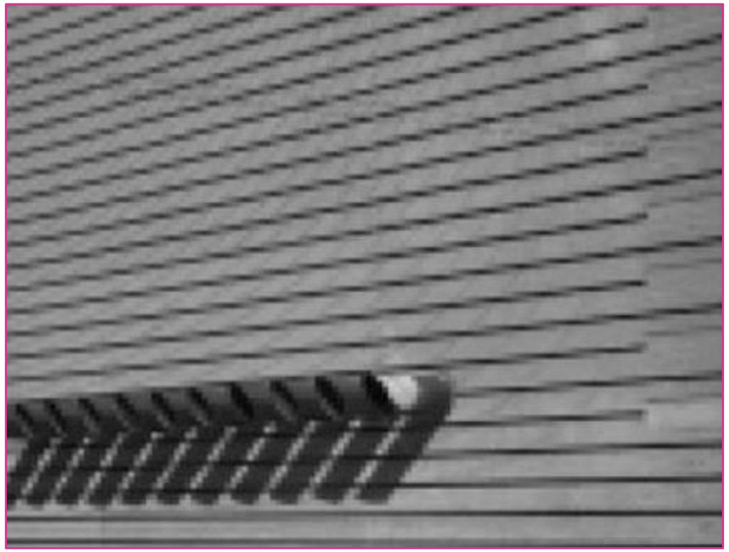
input



8x8 sq grid



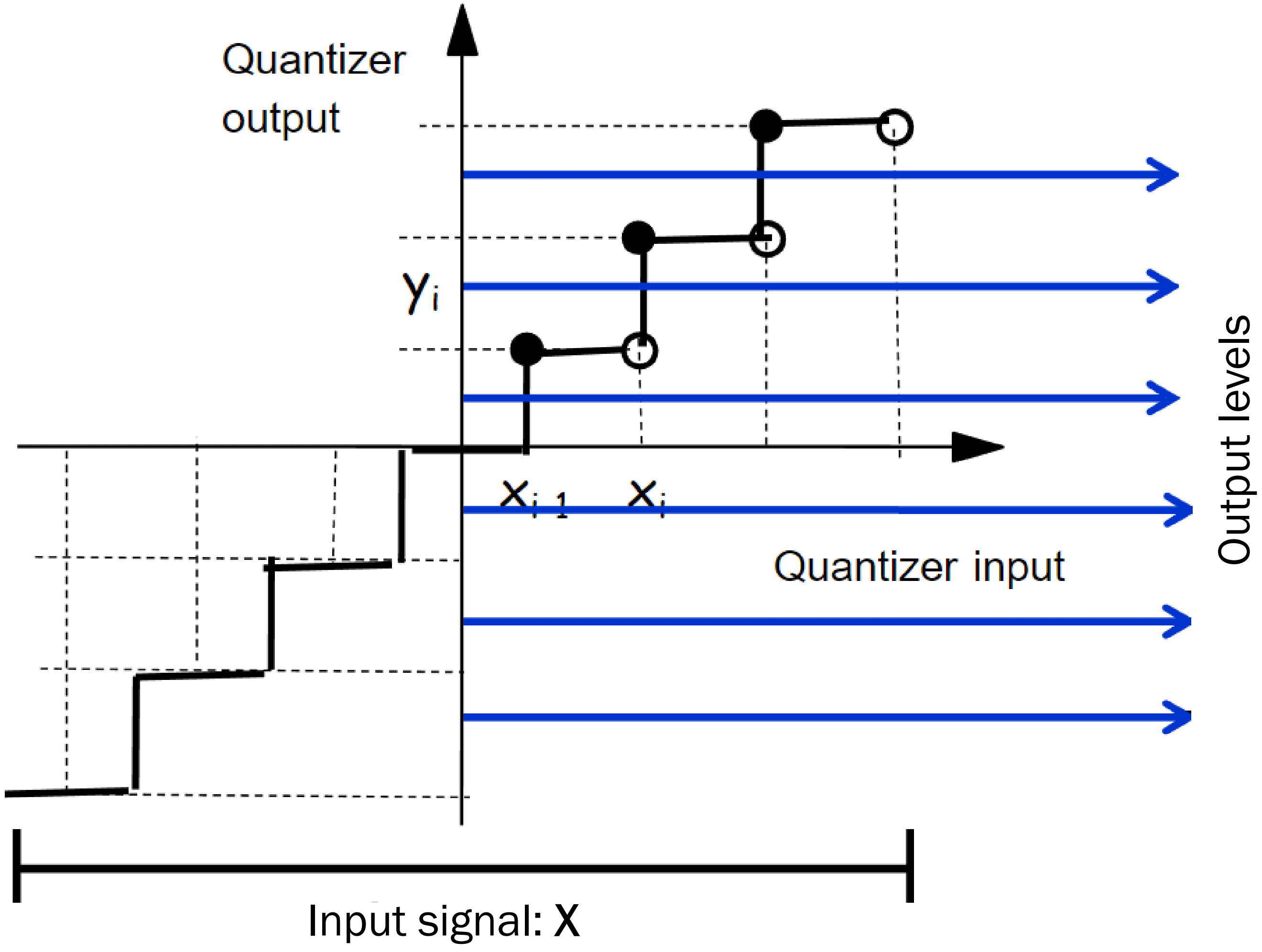
8x8 mean



Quantization

- Quantizer
  - SISO – scalar quantizer
  - mappings $[x_{i-1}, x_i) \rightarrow y_i$
  - what are the unknowns?

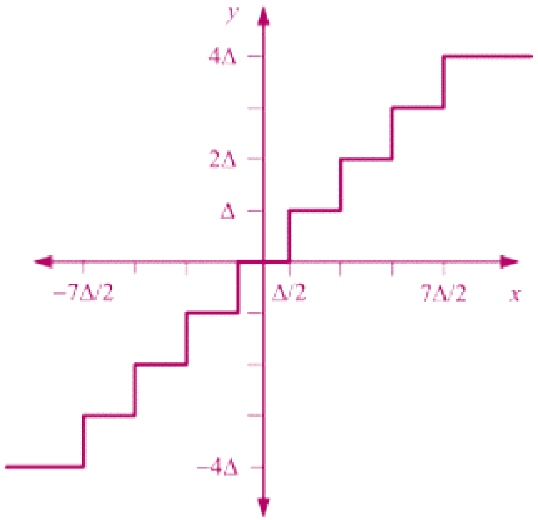
$$x \in [t_k, t_{k+1}) \Rightarrow Q(x) = r_k$$



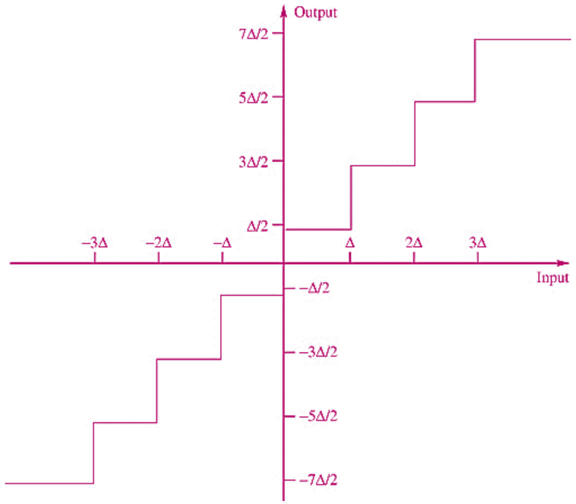
Quantization

- Uniform quantizers
  - all ranges divided equally with $\Delta = [t_k, t_{k+1})$ intervals
  - deadzone

Midtread quantizer



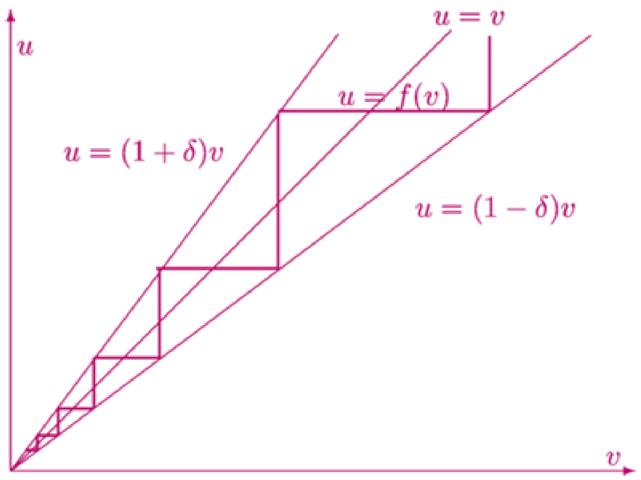
Midrise quantizer



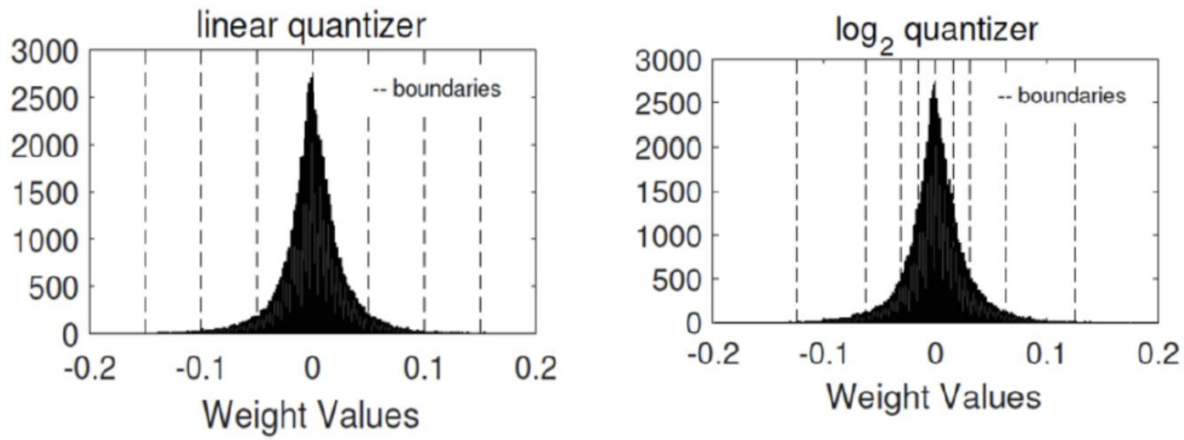
Quantization

- Non-uniform quantizers
  - ranges divided via predefined function which gives Δ intervals

Logarithmic quantizer



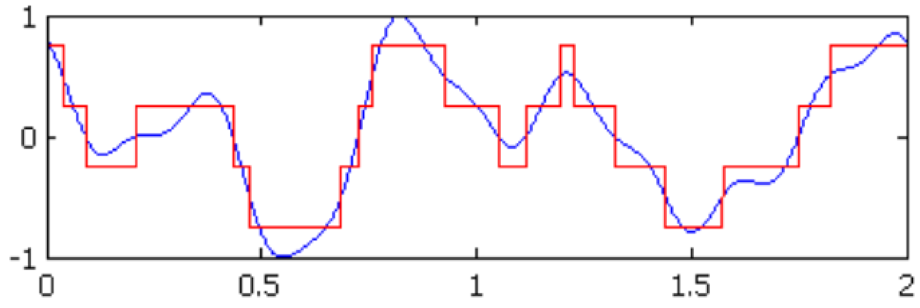
Logarithmic quantizer for image filter weights



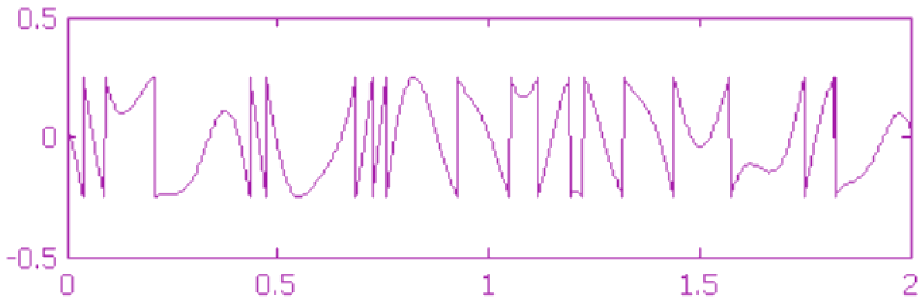
Quantization

- Non-uniform quantizers
  - can we design optimal quantizer?
  - optimal in the sense to minimize error (which error?)
  - input: $x_i \approx t_i$:thresholds
  - output: $y_i \approx r_i$:reconstructions
  - signal distribution is known: $p(x)$

Original and quantized signal



Quantization error



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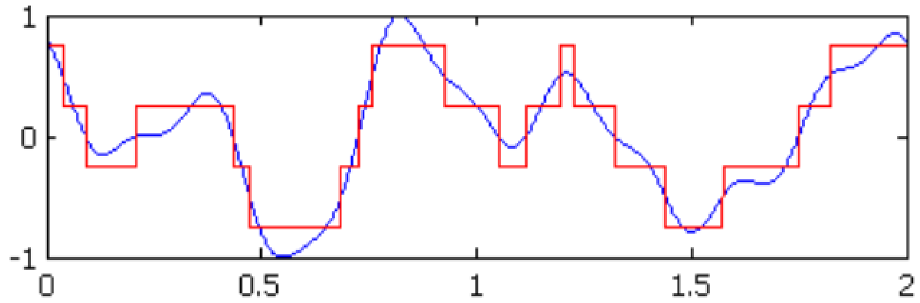
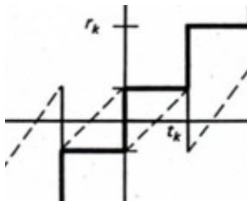
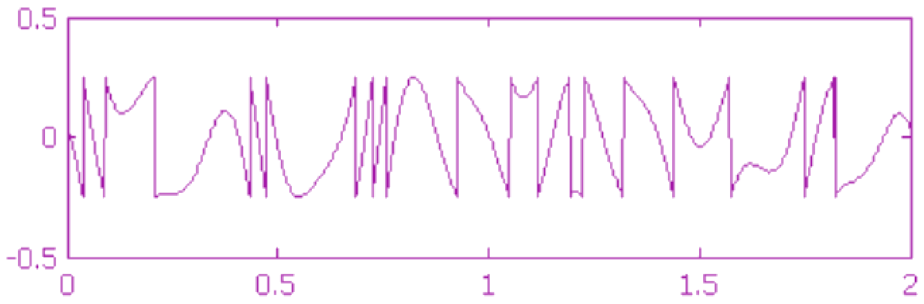


- MSE

$$D = \int_a^b (x - Q(x))^2 p(x)dx$$



Quantization error



sandhan@iitk.ac.in

Quantization

$$\begin{aligned} \underset{t_k, r_k}{\text{minimize}} \quad & D = \sum_{k=1}^{L} \int_{t_k}^{t_{k+1}} (x - r_k)^2 p(x) dx \\ \text{subject to} \quad & t_1 = a,\ t_{L+1} = b, \quad t_k < t_{k+1}, \\ & \quad t_k \le r_k \le t_{k+1} \quad k = 1, \ldots, L. \end{aligned}$$

Quantization

$$\underset{t_k, r_k}{\text{minimize}} \quad D = \sum_{k=1}^{L} \int_{t_k}^{t_{k+1}} (x - r_k)^2 p(x) dx$$

$$\text{subject to} \quad t_1 = a,\ t_{L+1} = b, \quad t_k < t_{k+1},$$
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$$\begin{aligned} \frac{\partial D}{\partial r_k} &= \frac{\partial}{\partial r_k} \sum_{j=1}^{L} \int_{t_j}^{t_{j+1}} (x - r_j)^2 p(x) dx \\ &= -2 \int_{t_k}^{t_{k+1}} (x - r_k) p(x) dx, \ \ k = 1, \ldots, L \end{aligned}$$

Quantization

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Fundamental theorem of calculus:
derivative with accumulation function
(c initial const., $f(t)$ is cts in open interval)

$$A(t) = \int_c^t f(x) dx$$

$$A'(t) = f(t)$$

Quantization

$$\underset{t_k, r_k}{\text{minimize}} \quad D = \sum_{k=1}^{L} \int_{t_k}^{t_{k+1}} (x - r_k)^2 p(x) dx$$

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$$= -(t_k - r_k)^2 p(t_k) + (t_k - r_{k-1})^2 p(t_k), \ k = 2, \ldots, I$$

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Leibniz integral rule:

Quantization

$$\underset{t_k, r_k}{\text{minimize}} \quad D = \sum_{k=1}^{L} \int_{t_k}^{t_{k+1}} (x - r_k)^2 p(x) dx$$

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$$A'(t) = f(t)$$

Leibniz integral rule: $$\frac{d}{dx}\left(\int_{a(x)}^{b(x)} f(x,t)\,dt\right) = f\big(x, b(x)\big) \cdot \frac{d}{dx} b(x) - f\big(x, a(x)\big) \cdot \frac{d}{dx} a(x) + \int_{a(x)}^{b(x)} \frac{\partial}{\partial x} f(x,t)\,dt$$

Quantization

$$\underset{t_k, r_k}{\text{minimize}} \quad D = \sum_{k=1}^{L} \int_{t_k}^{t_{k+1}} (x - r_k)^2 p(x) dx$$

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$$= -2 \int_{t_k}^{t_{k+1}} (x - r_k) p(x) dx,\ k = 1, \ldots, L$$

$$\frac{\partial D}{\partial t_k} = \frac{\partial}{\partial t_k} \sum_{j=1}^{L} \int_{t_j}^{t_{j+1}} (x - r_j)^2 p(x) dx$$
$$= -(t_k - r_k)^2 p(t_k) + (t_k - r_{k-1})^2 p(t_k),\ k = 2, \ldots, L$$

Assuming $p(x) > 0$ for each $x \in [a, b]$:

$$r_k = \frac{\int_{t_k}^{t_{k+1}} x p(x) dx}{\int_{t_k}^{t_{k+1}} p(x) dx},\ k = 1, \ldots, L$$
$$t_k = \frac{r_{k-1} + r_k}{2},\ k = 2, \ldots, L$$

Fundamental theorem of calculus:
derivative with accumulation function
(c initial const., $f(t)$ is cts in open interval)

$$A(t) = \int_c^t f(x) dx$$
$$A'(t) = f(t)$$

Quantization

- Lloyd-Max Quantizer
  - optimal MSE quantizer

$$r_k = \bar{x}_k = \frac{\int_{t_k}^{t_{k+1}} x p(x) dx}{\int_{t_k}^{t_{k+1}} p(x) dx}, \quad k = 1, \ldots, L \qquad \text{... (1)}$$

$$t_k = \frac{r_{k-1} + r_k}{2} = \frac{\bar{x}_{k-1} + \bar{x}_k}{2}, \quad k = 2, \ldots, L \qquad \text{... (2)}$$

Quantization

- Lloyd-Max Quantizer
  - optimal MSE quantizer

$$r_k = \bar{x}_k = \frac{\int_{t_k}^{t_{k+1}} x p(x) dx}{\int_{t_k}^{t_{k+1}} p(x) dx}, \quad k = 1, \ldots, L \quad \ldots (1)$$

$$t_k = \frac{r_{k-1} + r_k}{2} = \frac{\bar{x}_{k-1} + \bar{x}_k}{2}, \quad k = 2, \ldots, L \quad \ldots (2)$$

- Pseudo-code

: pick initial values for t (uniform grid)
: find r values using (1)
: find new t values using (2)
: repeat till both t, r converge

Quantization

- Properties of optimal quantizer
  - $$E[Q(x)] = \sum_k r_k p_k$$
    $$= E[x]$$
  - $$E[x - Q(x)] = 0$$

$$r_k = \frac{\int_{t_k}^{t_{k+1}} x p(x) dx}{\int_{t_k}^{t_{k+1}} p(x) dx}$$

$$p_k = \int_{t_k}^{t_{k+1}} p(x) dx$$

Quantization

- Properties of optimal quantizer
  - $$E[(x - Q(x))Q(x)] = \sum_k \int_{t_k}^{t_{k+1}} (x - r_k) r_k p(x) dx$$

$$r_k = \frac{\int_{t_k}^{t_{k+1}} x p(x) dx}{\int_{t_k}^{t_{k+1}} p(x) dx}$$

$$p_k = \int_{t_k}^{t_{k+1}} p(x) dx$$

Quantization

- Properties of optimal quantizer
  - $$E[(x-Q(x))Q(x)] = \sum_k \int_{t_k}^{t_{k+1}} (x-r_k) r_k p(x) dx$$
    $$r_k = \frac{\int_{t_k}^{t_{k+1}} xp(x)dx}{\int_{t_k}^{t_{k+1}} p(x)dx}$$
    $$= 0$$
    $$p_k = \int_{t_k}^{t_{k+1}} p(x)dx$$

Quantization

- Properties of optimal quantizer
  - $$E[(x - Q(x))Q(x)] = \sum_k \int_{t_k}^{t_{k+1}} (x - r_k) r_k p(x) dx$$

    $$= 0$$

    $$r_k = \frac{\int_{t_k}^{t_{k+1}} x p(x) dx}{\int_{t_k}^{t_{k+1}} p(x) dx}$$

    $$p_k = \int_{t_k}^{t_{k+1}} p(x) dx$$

  - error is uncorrelated with the quantizer's output

Quantization

- Lloyd-Max example

8 bpp

Quantization

- Lloyd-Max example

8 bpp

6 bpp

Quantization

- Lloyd-Max example

8 bpp

6 bpp

4 bpp

Conclusion

- Sampling
- Quantization

32x32

Conclusion

- Sampling
- Quantization

❑ Sampling

- Squares
- Hexagonal
- Aliasing

❑ Quantization

- Uniform
- Non-uniform
- Optimal

32x32